Hochster

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January 4, 2025

Chapter 1

Some Basic Facts

Lemma 1. *Let $h : A \to B$ be a homomorphism between commutative rings and suppose $O \subseteq Spec(A)$ is quasi-compact and open. Then $Spec(h)^{-1}(O)$ is quasi-compact.*

Proof. Note that if $D_a := \{p \in Spec(A) | a \notin p\}$ is a principal open subset of $Spec(A)$, where a is an element of A, then
$Spec(h)^{-1}(D_a)$
$= \{p \in Spec(B) | Spec(h)(p) \in D_a\}$
$= \{p \in Spec(B) | a \notin h^{-1}(p)\}$
$= \{p \in Spec(B) | h(a) \notin p\}$
$= D_{h(a)}$.

Now as O is quasi-compact and open, we can find a finite collection $\{D_{a_1}, ..., D_{a_n}\}$ of principal open subsets of $Spec(A)$ such that $O = \cup_{i=1}^{n} D_{a_i}$. Then
$Spec(h)^{-1}(O)$
$= Spec(h)^{-1}(\cup_{i=1}^{n} D_{a_i})$
$= \cup_{i=1}^{n} Spec(h)^{-1}(D_{a_i})$
$= \cup_{i=1}^{n} D_{h(a_i)}$,
which is a union of finitely many quasi-compact sets and so is itself quasi-compact (it is well-known that the principal open subsets of a prime spectrum is quasi-compact). □

Chapter 2

Proposition 2 of Hochster's Paper

Lemma 2. *Let n be a positive integer and let (X, A) be a spring. Suppose $a, b \in A$ satisfy $z(b) \subseteq z(a)$. Then $z(b^n) \subseteq z(a^n)$ and $(a\#b)^n = a^n\#b^n$.*

Lemma 3. *Let (f, h) be a spring morphism from (X, A) to (X', A'). Suppose $x \in X$ and $c \in A'$. Then $x \in d(h(c))$ if and only if $f(x) \in d(c)$.*

Proof. $x \in d(h(c))$ if and only if $h(c) \notin x$, which is true if and only if $c \notin f(x)$ (because $f = Spec(h)$), which is equivalent to $f(x) \in d(c)$. □

Proposition 4. *Let $A = (X, A)$ and $A^{'} = (X', A')$ be springs with indices v and v' respectively. Let (f, h) be an indexed spring homomorphism from A to $A^{'}$. If (a, b) belongs to $G(A^{'}, v')$, then (h(a),h(b)) belongs to $G(A, v)$.*

Proof. We first show that $z(h(b)) \subseteq z(h(a))$. Indeed, if $x \in X$ belongs to $z(h(b))$, then by Lemma 3 we have that $f(x) \in z(b)$. As $z(b) \subseteq z(a)$, $f(x) \in z(a)$. Then Lemma 3 implies that $x \in z(h(a))$.

By Theorem 3 of Hochster's paper, we now only need to show that for any element $p = (y, x)$ of $\sigma(X)$ such that $h(a)(y) \neq 0$ it is true that $v_p(h(b)) \leq v_p(h(a))$, with equality only if $h(b)(x) \neq 0$.

Indeed, if $y \in d(h(a))$, then by Lemma 3 we know that $f(y) \in d(a)$. Because $(f(y), f(x)) \in \sigma(X')$, Theorem 3 in Hochster's thesis implies that $v_{f(p)}(a) \geq v_{f(p)}(b)$. By the definition of an indexed spring morphism, $v_{f(p)}(a) = v_p(h(a))$ and $v_{f(p)}(b) = v_p(h(b))$. Hence $v_p(h(a)) \geq v_p(h(b))$.

If $v_p(h(b)) = v_p(h(a))$, then $v_{f(p)}(a) = v_{f(p)}(b)$ and so $f(x) \in d(b)$. Thus, by Lemma 3 we have that $x \in d(h(b))$ and we are done! □

Proposition 5. *Let $A = (X, A)$ and $A^{'} = (X', A')$ be springs with indices v and v' respectively. Let (f, h) be an indexed spring morphism from A to $A^{'}$. Suppose $(a, b) \in G(A^{'}, v')$. Then there exists a unique ring homomorphism h_1 from $A'[a\#b]$ to $A[h(a), h(b)]$ that extends h and maps $a\#b$ to $h(b)\#h(b)$.*

Proof. The most difficult part of the proof is to show that if $r = q(a\#b) \in A'$ where $q = \Sigma_{i=0}^{m} a_i t^i$, then $r' := \Sigma_{i=0}^{m} h(a_i)(h(a)\#h(b))^i$ belongs to A and equals $h(r)$. The purpose of showing this is to guarantee that if we construct the homomorphism h_1 in the obvious way, then h_1 is well-defined.

Let $c := b^m r$. Then $h(b^m)r' = \Sigma_{i=0}^m h(a_i)h(a)^i h(b)^{m-i} = h(b^m)h(r)$. We then try to prove that $r'(x) = h(r)(x)$ for any $x \in X$. We show this by classifying the x's. If $x \in d(h(b))$, then it is clear that $r'(x) = h(r)(x)$ because we have shown that $h(b^m)r' = h(b^m)h(r)$.

Now assume $x \in z(h(b))$. Let $\overline{h} : \frac{A'}{h^{-1}(x)} \to \frac{A}{x}$ be the ring homomorphism induced by h. Then
$r'(x) = h(a_0)(x) = h(a_0) + x(\in \frac{A}{x}) = \overline{h}(a_0 + h^{-1}(x))$
$= \overline{h}(a_0(f(x))) = \overline{h}(r(f(x))) = \overline{h}(r + h^{-1}(x))$
$= h(r) + x(\in \frac{A}{x}) = h(r)(x)$.

It is easy to show that the obvious way to define h_1 satisfy the axioms of ring homomorphisms, and that this extension of h is unique. □

Proposition 6. *Let $A = (X, A)$ and $A^{'} = (X', A')$ be springs with indices v and v' respectively. Let (f, h) be an indexed spring morphism from (X, A) to (X', A'). Suppose $(a, b) \in G(A^{'}, v')$ and let h_1 be the ring homomorphism constructed in Proposition 5. Then (f, h_1) is an indexed spring morphism from $A[h(a)\#h(b)], v$ to $A^{'}[a\#b], v'$.*

Proof. Let $r = q(a\#b)$ where $q = \Sigma_{i=0}^m a_i t^i \in A'[t]$. We first show that $f^{-1}(z(r)) = z(h_1(r))$. By a previous comment, this is sufficient for proving that (f, h_1) is a spring morphism from $A[h(a)\#h(b)]$ to $A^{'}[a\#b]$, as r has been taken arbitrarily.

Pick some $x \in X$. Let $c := b^m r = \Sigma_{i=0}^m a_i a^i b^{m-i}$. Then $h(c) = \Sigma_{i=0}^m h(a_i)h(a)^i h(b)^{m-i} = h(b)^m h_1(r)$ and we have:
$x \in f^{-1}(z(r))$
$\Leftrightarrow f(x) \in z(r) = (z(c) \cap d(a)) \cup (z(a_0) \cap z(a))$
$\Leftrightarrow x \in (z(h(c)) \cap d(h(a))) \cup (z(h(a_0)) \cap z(h(a)))$
$\Leftrightarrow x \in z(h_1(r))$.

We still need to show that for any $p = (y, x) \in \sigma(X)$ with $y \in d(h_1(r))$ it is true that $v_p(h_1(r)) = v_{f(p)}(r)$. Indeed, because $h(c) = h(b)^m h_1(r)$, $v_p(h_1(r)) = v_p(h(c)) - v_p(h(b^m)) = v_{f(p)}(c) - v_{f(p)}(b^m) = v_{f(p)}(r)$. □

Proposition 7. *Let $A = (X, A)$ and $A^{'} = (X', A')$ be springs with indices v and v' respectively. Let (f, h) be an indexed spring morphism from (X, A) to (X', A'). Suppose $(a, b) \in G(A^{'}, v')$, $A^{''}$ and $A^{'''}$ extend A and $A^{'}[a\#b]$ respectively, and $h_2 : A''' \to A''$ is a ring homomorphism extending h. If (f, h_2) is a spring morphism from $A^{''}$ to $A^{'''}$, then $h(a)\#h(b) = h_2(a\#b)$, meaning that $h(a)\#h(b)$ belongs to A''.*

Proof. We show that for any $x \in X$, $h_2(a\#b)(x) = (h(a)\#h(b))(x)$. Indeed, if $h(b)(x) \neq 0$, then as
$h_2(a\#b)h(b) = h_2(a\#b)h_2(b) = h_2(a) = h(a) = (h(a)\#h(b))h(b)$,
we immediately have that $h_2(a\#b)(x) = (h(a)\#h(b))(x)$. If $h(b)(x) = 0$, then $x \in z(h_2(b))$ and so $f(x) \in z(b)$ by Lemma 3. This means $(a\#b)(f(x)) = 0$. Then Lemma 3 implies $h_2(a\#b)(x) = 0$, so $h_2(a\#b)(x) = (h(a)\#h(b))(x)$. □